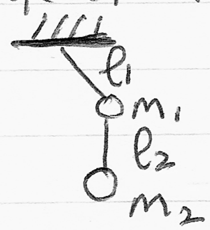
Linear Chains, etc.

→ Small Oscillations II – { Chains, Strings and the Transition Discrete → Continuous

– previously, considered few-degree-of-freedom systems, i.e.



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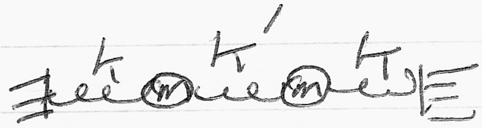
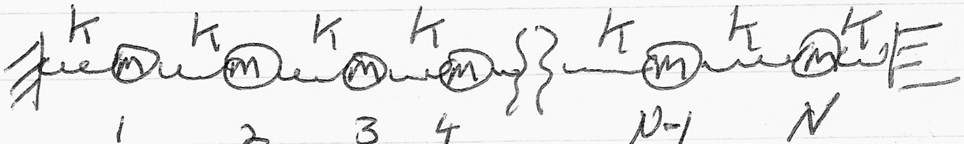


now, consider systems with $N \gg 1$ degrees of freedom, i.e.

(separated by l at equilibrium)

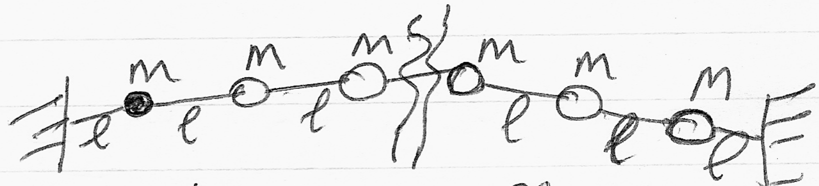
i) – linear chain (1D oscillators)



(application → solid)

(identical components) → monatomic

ii) – massless string (loaded)



uniform tension τ

uniform { mass m, separation l }

For i)

$$L = \sum_{i=1}^{N} \left(\frac{1}{2} m \dot{x}_i^2 - \left(\frac{1}{2} k (x_i - x_{i-1})^2 + \underbrace{\frac{1}{2} k (x_{i+1} - x_i)^2} \right) \right)$$

$\frac{1}{2} k (x_{i+1} - x_i)^2$

$$\begin{cases} x_0 \equiv 0 \\ x_{N+1} \equiv 0 \end{cases}$$

or simply

$$L = \sum_{i=1}^{N} \left(\frac{1}{2} m \dot{x}_i^2 - \frac{k}{2} (x_{i+1} - x_i)^2 \right)$$

{ Compressional Modes

For ii),

$$L = \sum_{i=1} \left(\frac{1}{2} m \dot{x}_i^2 - \frac{\tau}{2\ell} (y_{i+1} - y_i)^2 \right)$$

{ transverse modes

≈ identical systems.

∴ Hereafter, focus on i.)

motivations for i.)
- monatomic chain is simplest example of elastic wave in solid
- step toward continuous system
  i.e. now discrete → masses separated by ℓ

Proceeding:

$$m \ddot{x}_i - k \left[(x_{i+1} - x_i) + (x_{i-1} - x_i) \right] = 0$$

$$\ddot{x}_i + \frac{k}{m} \left[2x_i - (x_{i+1} + x_{i-1}) \right] = 0$$

$$x_i = \hat{x}_i e^{-i\omega t}$$

$$\left(\frac{2k}{m} - \omega^2\right)\hat{x}_i - \frac{k}{m}\left(\hat{x}_{i-1} + \hat{x}_{i+1}\right) = 0$$

For eigenvalues, $\det \underline{\underline{A}} = 0$

$$\underline{\underline{A}} = \begin{vmatrix} \frac{2k}{m} - \omega^2 & -k/m & & & 0 \\ -k/m & \frac{2k}{m} - \omega^2 & -k/m & & \\ & -k/m & \frac{2k}{m} - \omega^2 & -k/m & \\ 0 & & -k/m & \frac{2k}{m} - \omega^2 & -k/m \end{vmatrix}$$

i.e. A tri-diagonal.

Now, taking masses separated by ℓ, take

$$\hat{x}_n \sim e^{i(n\ell)\alpha} \qquad \begin{cases} n \equiv \text{bead \#} \\ \alpha \equiv \text{wave \#} \\ \ell \equiv \text{spacing} \end{cases}$$

$k x$ ↑ ; ↓ wave-vector

$$\Rightarrow \left(\frac{2k}{m} - \omega^2\right)e^{i[i\,\ell\alpha]} - \frac{k}{m}\left(e^{i[(i+1)\ell\alpha]} + e^{i[(i-1)\ell\alpha]}\right) = 0$$

careful i's.

$$\therefore \left(\frac{2k}{m} - \omega^2\right) - \frac{2k}{m}\cos\left[\alpha\ell\right] = 0$$

Note: says $\hat{x}_{n+m} = e^{i m \ell \alpha}\,\hat{x}_m$

phase displ $\sim m\ell$

Sol/

$$\omega^2 = \frac{2k}{m}(2)\left[\frac{1-\cos(\alpha \ell)}{2}\right]$$

$$= \frac{4k}{m}\sin^2\left(\frac{\alpha \ell}{2}\right)$$

$$\Rightarrow \boxed{\omega^2 = \frac{4k}{m}\sin^2(\alpha\ell/2)}$$

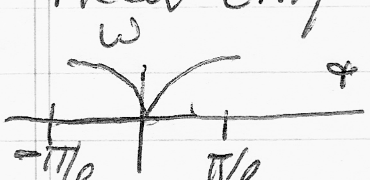
$$\omega = 2\sqrt{k/m}\,|\sin \alpha\ell/2|$$

Note:

① – $\omega = \omega_{max}\,|\sin \ell\alpha/2|$; $\omega_{max}^2 = 4k/m$

$$\begin{cases} \omega(\alpha) = \omega(-\alpha) \\ \alpha' = \alpha + 2\pi/\ell \quad \text{leaves } \omega \text{ invariant} \end{cases}$$

i.e. need only define α on $\left[-\pi/\ell,\ \pi/\ell\right]$



& repeats

i.e. First Brillouin Zone, only needed

② – for $\ell\alpha/2 \ll 1$

i.e. wavelength $\alpha^{-1} >>$ bead spacing ℓ

$\Rightarrow$ continuum limit

Then $\omega = \sqrt{k/m}\,\ell\,\alpha$

$= \alpha\left[\ell\sqrt{k/m}\right]$

akin to acoustic wave

$\omega = k\, c_s$

$$\begin{cases} k \leftrightarrow \alpha \\ c_s \leftrightarrow \ell\sqrt{k/m} \\ \dfrac{\gamma p}{\rho} \leftrightarrow \dfrac{\ell^2 k}{m} \end{cases}$$

$\ell^2 k \rightarrow$ stored elastic energy (springiness)

$m \rightarrow$ inertia

③ — observe maximum frequency propagated is:

$\omega^2 = \omega_{max}^2 = 4k/m$ i.e. $\begin{cases} \omega^2 > \omega_{max}^2 & \text{not propagated} \\ \omega^2 < \omega_{max}^2 & \text{propagated} \end{cases}$

Chain acts as low-pass filter
Higher frequencies evanescent!

④ — for propagation structure;

$\omega = 2\sqrt{k/m}\left[\sin(\alpha\ell/2)\right]$

$V_{gr} = d\omega/d\alpha = \ell\sqrt{k/m}\cos(\alpha\ell/2)$

i.e. $V_{gr} \simeq \ell\sqrt{k/m} \sim c_{s\,eff}$ for $\alpha\ell \ll 1$
(a la' sound)

but $\lim_{\alpha \to \pi/\ell} v_{gr} = \ell\sqrt{k/m}\,\cos(\pi/2) \to 0$

i.e. modes at edge of Brillouin zone non-propagating

modes in middle of zone propagate at acoustic speed.

Can also observe that:

$$x_{i+1} + x_{i-1} - 2x_i = e^{i[\alpha\ell]i}\left(e^{i\alpha\ell} + e^{-i\alpha\ell} - 2\right)$$

$$= 2e^{i[i\alpha\ell]}\left(\cos\alpha\ell - 1\right)$$

so $\cos\alpha\ell / 1 \sim$ ratio of $(x_{i+1} + x_{i-1})/2x_i$

$\sim$ mean phase ratio

so $\alpha\ell \ll 1$ (in zone) $\Rightarrow$ neighbors on chain vibrate in phase $\to$ propagation

$\cos = 1$

$\alpha\ell \sim \pi$ (zone boundary) $\Rightarrow$ neighbors on chain vibrate out of phase $\to$ no propagation

$\cos = -1$

What if?:

→ Boundary Conditions

Can distinguish 2 cases: periodic B.C.'s; fixed end B.C.'s.

1) Periodic B.C.'s

Now, $x_i = Ae^{i[i\ell\alpha]}$

notational clarity ⇒ $x_n = Ae^{i[n\ell\alpha]}$

$$1 < n < N.$$

For periodic B.C.'s,

$$x_n = x_{n+N} \Rightarrow e^{iN\ell\alpha} = 1$$

→ mode index.

$$\therefore \quad N\ell\alpha = 2\pi p$$

$$\Rightarrow \boxed{\alpha = \frac{2\pi p}{N\ell}} \qquad p = \begin{cases} 0, \pm 1, \dots \pm \frac{1}{2}(N-1) & N \text{ odd} \\ 0, \pm 1, \dots \frac{1}{2}N & N \text{ even} \end{cases}$$

Note: guarantees N normal modes.

2) Fixed end B.C.'s: $\left.\begin{array}{l} x_0 = 0 \\ x_{N+1} = 0 \end{array}\right\}$ guarantees ends fixed

$\Rightarrow \quad x_0 = x_{N+1} = 0$

$$x_n = Ae^{in\alpha\ell} + Be^{-in\alpha\ell}$$

$$= A\sin(n\ell\alpha) + B\cos(n\ell\alpha)$$

$B = 0 \rightarrow n = 0$ ✓

$(N+1)\alpha\ell = p\pi \quad ; \quad p = 1, \ldots N$

mode index

$\Rightarrow$

$$\boxed{\alpha_p = p\pi/\ell(N+1)}$$

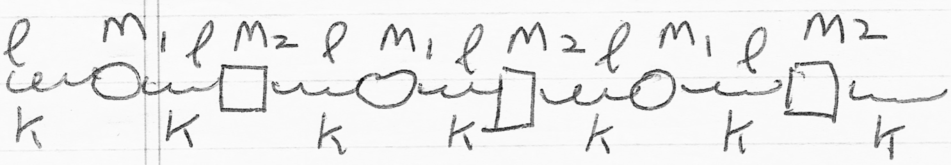
{acts to quantize k

$$\therefore \quad x_n(t) = A_n \sin\left(\frac{n\cancel{\ell}p\pi}{\cancel{\ell}(N+1)}\right) e^{-i\omega_p t}$$

where $\omega_p^2 = \frac{4k}{m}\sin^2\left(\frac{p\pi\cancel{\ell}}{2\cancel{\ell}(N+1)}\right)$

- Diatomic Chain

→ consider slightly richer toy model, namely the diatomic chain



{un-equal masses!

then, no loss of generality to associate

$m_1 \rightarrow x_{2n}$ (evens)
$m_2 \rightarrow x_{2n+1}$ (odds) } positions

∴ can immediately write dynamical equations

$$m_1 \ddot{x}_{2n} = -k\,(2x_{2n} - x_{2n-1} - x_{2n+1})$$

$$m_2 \ddot{x}_{2n+1} = -k\,(2x_{2n+1} - x_{2n} - x_{2n+2})$$

solution of form:

$$x_{2n} = A e^{i2n\ell\alpha} e^{-i\omega t} \qquad \text{(evens)}$$

$$x_{2n+1} = B e^{i(2n+1)\ell\alpha} e^{-i\omega t} \qquad \text{(odds)}$$

(consider one mass → 0, ∞)

$$-m_1 \omega^2 A = -k\left(2A - \left(e^{il\alpha} + e^{-il\alpha}\right) B\right)$$

$$-m_2 \omega^2 B = -k\left(2B - \left(e^{il\alpha} + e^{-il\alpha}\right) A\right)$$

$\Rightarrow$

$$(-m_1\omega^2 + 2k) A - k(2\cos l\alpha)\ B = 0$$

$$(-2k\cos l\alpha)A + (-m_2\omega^2 + 2k) B = 0$$

$\therefore$

$$\boxed{(\omega^2 - 2k/m_1)(\omega^2 - 2k/m_2) - \frac{4k^2}{m_1 m_2}\cos^2 l\alpha = 0}$$

$\Rightarrow$ dispersion relation:

$$\omega^2 = k\left(\frac{1}{m_1} + \frac{1}{m_2}\right) \pm k\left\{\left(\frac{1}{m_1} + \frac{1}{m_2}\right)^2 - \frac{4\sin^2(\alpha l)}{m_1 m_2}\right\}^{1/2}$$

$$1/\mu \equiv 1/m_1 + 1/m_2$$ $\Rightarrow$ reduced mass, as usual.

$$\omega^2 = k/\mu \pm k/\mu\left\{1 - \frac{4\mu^2}{m_1 m_2}\sin^2(\alpha l)\right\}^{1/2}$$

$\therefore$ dispersion relation:

$$\boxed{\omega^2 = \frac{k}{\mu}\left\{1 \pm 1\left\{1 - \frac{4\mu^2}{m_1 m_2}\sin^2(\alpha l)\right\}^{1/2}\right\}}$$

Can immediately observe:

→ system supports 2 modes

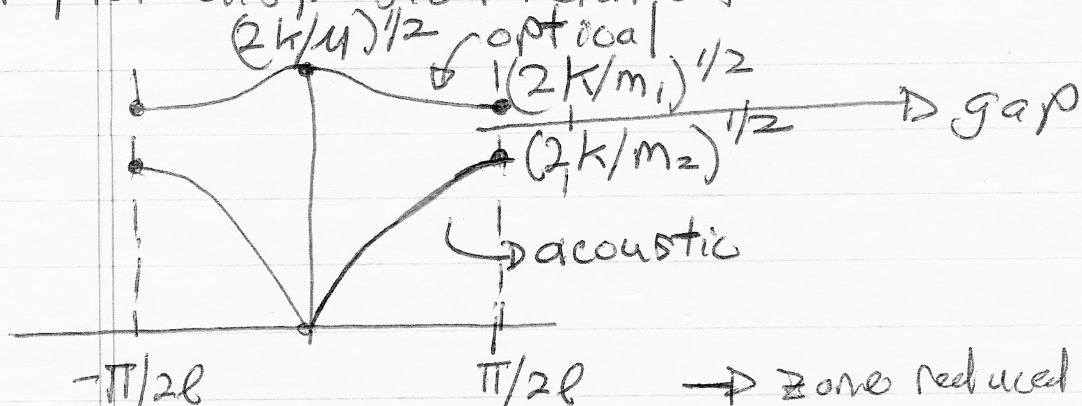
- low frequency → "acoustic" mode (ala' sound)

→ analogous to mode of monatomic chain

- high frequency → "optical" mode (ala' plasma) (vibration)

→ new

Can plot dispersion relation



→ zone reduced by 2! (scale)

Note: - acoustic mode $\omega \sim \alpha \left(\frac{kl^2}{m_2+m_1}\right)^{1/2}$

$\sim k c_{s_{eff}}$

as $kl \to 0$ ⇒ mass neighbors vibrate in phase

$x_n = x_{n+1}$

solid → phonon ($\omega = k c_s$)

optical mode $\omega \sim (2k/M)^{1/2}$

as $kl \to 0$; $m_1 x_n + m_2 x_{n+1} = 0$

i.e. neighboring masses vibrate out of phase, weighted by masses

solid $\to$ analogous collective mode is EM wave
$\omega^2 = \omega_{pe}^2 + c^2 k_1^2$ or plasmon
$\omega^2 = \omega_{pe}^2 + k^2 v_{Te}^2$

i.e. $k \to 0$, frequency constant!

$\to$ Note gap $\to$ no propagation for

$$(2k/m_2)^{1/2} < \omega < (2k/m_1)^{1/2}$$

$\to$ consequence of fact
phonon $\leftrightarrow$ inertia of heavy mass
optical $\leftrightarrow$ inertia of light mass (in ω_p^2)

→ Transition to Continuum

To recover continuum $\left\{ \underline{\text{i.e.}} \begin{array}{l} \text{elastic medium} \\ \text{massive string} \end{array} \right.$

take $\quad N \to \infty \quad$ $\underline{\underline{\text{with}}}$ $\underline{\underline{\text{constant}}}$ $\quad L = (N+1)\ell$

$\quad\quad \ell \to 0$

$$\left\{ \begin{array}{l} \dfrac{m}{\ell} = \mu = \text{const.} \\ K\ell = \mathcal{K} = \text{const.} \end{array} \right.$$

Note: "$N \to \infty$" means $N > p$, for all modes p.

Then;

$$\omega_p^2 = \frac{4K}{m} \sin^2\left(\frac{p\pi}{2(N+1)}\right)$$

$$\cong \frac{4K}{m}\left(\frac{p\pi}{2(N+1)}\right)^2$$

$$= \frac{(p\pi)^2}{[(N+1)\ell]^2} \frac{K\ell^2}{m}$$

$$= \left(\frac{p\pi}{L}\right)^2 \left(\frac{\mathcal{K}}{\mu}\right) = \left(\frac{p\pi}{L}\right)^2 c_s^2$$

$$c_s^2 = K\ell^2/m = (K\ell)\,\ell/m = \mathcal{K}/\mu$$

$$\to \quad \omega^2 = k^2 c_s^2 \quad ; \quad \begin{array}{l} c_s^2 = \mathcal{K}/\mu \\ k = p\pi/L \end{array}$$